A hidden non-Abelian monopole in a 16-dimensional isotropic harmonic oscillator

2009 J. Phys. A: Math. Theor. 42 175204

(http://iopscience.iop.org/1751-8121/42/17/175204)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 175204 (8pp) doi:10.1088/1751-8113/42/17/175204

A hidden non-Abelian monopole in a 16-dimensional isotropic harmonic oscillator

Van-Hoang Le, Thanh-Son Nguyen and Ngoc-Hung Phan

Department of Physics, HCMC University of Pedagogy, 280 An Duong Vuong, Ward 10, Dist. 5, Ho Chi Minh City, Vietnam

Received 13 January 2009, in final form 12 March 2009
Published 1 April 2009
Online at stacks.iop.org/JPhysA/42/175204

Abstract

We suggest one variant of generalization of the Hurwitz transformation by adding seven extra variables that allow an inverse transformation to be obtained. Using this generalized transformation we establish the connection between the Schrödinger equation of a 16-dimensional isotropic harmonic oscillator and that of a nine-dimensional hydrogen-like atom in the field of a monopole described by a septet of potential vectors in a non-Abelian model of 28 operators. The explicit form of the potential vectors and all the commutation relations of the algebra are given.

PACS numbers: 03.65.−w, 03.65.Fd

1. Introduction

There exists a connection between an n-dimensional hydrogen-like atom and an N-dimensional harmonic oscillator for very few cases of dimension $n \to N$ such as $2 \to 2, 3 \to 4,\ 5 \to 8, 9 \to 16$ (see, for example [1–4]). Indeed, according to the Hurwitz theorem [5], only for the cases of dimension mentioned above can one establish a bilinear transformation from n-dimensional real space to N-dimensional real space that satisfies the Euler identity

$$\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = u_1^2 + u_2^2 + \cdots + u_N^2.$$

These bilinear transformations are called Levi–Civita [6], Kustaanheimo–Stiefel [7], Hurwitz transformations (see, for example [8]) depending on the cases of dimension $(2 \to 2)$, $(3 \to 4)$, $(5 \to 8, 9 \to 16)$ correspondingly. Using them with $(N - n)$ certain constraints applied to the wavefunctions one can transform the Schrödinger equation of an N-dimensional harmonic oscillator to that of a hydrogen-like atom in n-dimensional real space [1–4].

In the case of $3 \to 4$, the inversion of Kustaanheimo–Stiefel transformation $u_1, u_2, u_3, u_4 \to x_1, x_2, x_3, \phi$ has been established by introducing the extra variable $\phi = \mathrm{artan}(u_2/u_1)$ [9]. This inverse transformation is called Cayley–Klein transformation [9] and was used in the works [10] to connect a harmonic oscillator not only to a hydrogen-like atom but also to it with the presence of the Dirac monopole field. For more interpretations of the extra variable ϕ see also [11].

In [12] we have introduced three Euler angles as extra variables to Hurwitz transformation ($5 \to 8$) that allow the inverse transformation from $u_1, u_2, \ldots, u_8$ to $x_1, x_2, \ldots, x_5, \phi_1, \phi_2, \phi_3$ to be established. Later on, the result was used for the connection between the eight-dimensional harmonic oscillator and the problem of a five-dimensional hydrogen atom in the field of the SU(2) non-Abelian monopole [13]. This problem has attracted a great deal of interest [14, 15].

Naturally, the question arising is how to apply a similar concept for the last case of Hurwitz transformation ($9 \to 16$). In the present paper, we will generalize the Hurwitz transformation by adding to it seven extra variables that allow us to build an inverse transformation $u_1, u_2, \ldots, u_{16} \to x_1, x_2, \ldots, x_9, \phi_1, \phi_2, \ldots, \phi_7$. Applying the obtained result we lead the Schrödinger equation of a 16-dimensional harmonic oscillator to that of a nine-dimensional system of 'charge-dyon' in a non-Abelian model of closed algebra of 28 elements. This system looks like a nine-dimensional hydrogen-like atom in a field of a monopole described by septet of potential vectors. In other words, we have found the non-Abelian monopole solution hidden in the solutions of a 16-dimensional harmonic oscillator.

2. Generalized Hurwitz transformation ($9 \to 16$)

According to [4], the Hurwitz transformation can be written as follows:

$$x_k = 2(\Gamma_k)_{st} u_s v_t \qquad x_9 = u_s u_s - v_s v_s \tag{1}$$

that connects the nine-dimensional real space of coordinates $x_1, x_2, \ldots, x_9$ to the 16-dimensional real space of coordinates $u_1, u_2, \ldots, u_8, v_1, v_2, \ldots, v_8$. Indices k, s, t in (1) run from 1 to 8. Here, repeating indices means summation over them; matrices Γ_k are defined as follows:

$$\Gamma_1 = \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} \beta\alpha_1\alpha_3 & 0 \\ 0 & \beta\alpha_1\alpha_3 \end{bmatrix}, \qquad \Gamma_3 = \begin{bmatrix} \alpha_3 & 0 \\ 0 & \alpha_3 \end{bmatrix},$$
$$\Gamma_4 = \begin{bmatrix} \alpha_1 & 0 \\ 0 & -\alpha_1 \end{bmatrix}, \qquad \Gamma_5 = \begin{bmatrix} 0 & -\mathrm{i}\alpha_1\alpha_2 \\ \mathrm{i}\alpha_1\alpha_2 & 0 \end{bmatrix}, \qquad \Gamma_6 = \begin{bmatrix} 0 & -\mathrm{i}\beta\alpha_2\alpha_3 \\ \mathrm{i}\beta\alpha_2\alpha_3 & 0 \end{bmatrix}, \tag{2}$$
$$\Gamma_7 = \begin{bmatrix} 0 & -\beta\alpha_3 \\ \beta\alpha_3 & 0 \end{bmatrix}, \qquad \Gamma_8 = \begin{bmatrix} 0 & \alpha_1 \\ \alpha_1 & 0 \end{bmatrix}$$

in which $\beta = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$, $\alpha_k = \left[\begin{smallmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{smallmatrix}\right]$; I and σ_k ($k = 1, 2, 3$) are unit and Pauli matrices, respectively. All matrices (2) are real, either symmetrical or anti-symmetrical and satisfy the condition

$$\Gamma_k \Gamma_j^T - \Gamma_j \Gamma_k^T = 2\delta_{jk} I. \tag{3}$$

The Euler identity

$$r = \sqrt{x_\lambda x_\lambda} = u_s u_s + v_s v_s \tag{4}$$

can be easily verified. From here on, Greek symbols are used for indicating the coordinates x_λ with index $\lambda = 1 \to 9$ but wherever necessary we use the Latin symbol to denote the index of x_j with value $j = 1 \to 8$.

Now let us define also seven extra variables as follows:

$$\begin{aligned}
&\alpha_1 = \arctan\frac{u_2}{u_1}, \quad \alpha_2 = \arctan\frac{u_4}{u_3}, \quad \alpha_3 = \arctan\frac{u_6}{u_5}, \quad \alpha_4 = \arctan\frac{u_8}{u_7},\\
&\phi_1 = \arctan\frac{2\sqrt{u_1^2+u_2^2+u_3^2+u_4^2}\sqrt{u_5^2+u_6^2+u_7^2+u_8^2}}{u_1^2+u_2^2+u_3^2+u_4^2-u_5^2-u_6^2-u_7^2-u_8^2},\\
&\phi_2 = \arctan\frac{2\sqrt{u_1^2+u_2^2}\sqrt{u_3^2+u_4^2}}{u_1^2+u_2^2-u_3^2-u_4^2},\\
&\phi_3 = \arctan\frac{2\sqrt{u_5^2+u_6^2}\sqrt{u_7^2+u_8^2}}{u_5^2+u_6^2-u_7^2-u_8^2}.
\end{aligned} \tag{5}$$

We see that all additional variables defined by (5) can be interpreted as angles.

Formula (1) with additional (5) is one variant of generalization of Hurwitz transformation which connects two 16-dimensional real spaces $(x, \phi, \varphi) \to (uv)$. Especially, the inverse transformation $(uv) \to (x, \phi, \varphi)$ can be established now as follows:

$$u_s = \sqrt{r + x_9}\, b_s(\phi\alpha) \qquad v_s = \frac{x_j}{\sqrt{r + x_9}} H_{js}(\phi\alpha), \tag{6}$$

where the functions $b_s(\phi\alpha)$ depend on angles (5) only and have the explicit form

$$\begin{aligned}
b_1 &= \tfrac{1}{\sqrt{2}}\cos(\phi_1/2)\cos(\phi_2/2)\cos\alpha_1, & b_2 &= \tfrac{1}{\sqrt{2}}\cos(\phi_1/2)\cos(\phi_2/2)\sin\alpha_1,\\
b_3 &= \tfrac{1}{\sqrt{2}}\cos(\phi_1/2)\sin(\phi_2/2)\cos\alpha_2, & b_4 &= \tfrac{1}{\sqrt{2}}\cos(\phi_1/2)\sin(\phi_2/2)\sin\alpha_2,\\
b_5 &= \tfrac{1}{\sqrt{2}}\sin(\phi_1/2)\cos(\phi_3/2)\cos\alpha_3, & b_6 &= \tfrac{1}{\sqrt{2}}\sin(\phi_1/2)\cos(\phi_3/2)\sin\alpha_3,\\
b_7 &= \tfrac{1}{\sqrt{2}}\sin(\phi_1/2)\sin(\phi_3/2)\cos\alpha_4, & b_8 &= \tfrac{1}{\sqrt{2}}\sin(\phi_1/2)\sin(\phi_3/2)\sin\alpha_4.
\end{aligned}$$

Matrix elements $H_{js}(\phi\alpha)$ can be expressed via $b_s(\phi\alpha)$ as

$$H(\beta\alpha) = \begin{pmatrix}
b_1 & -b_2 & b_3 & b_4 & -b_5 & -b_6 & b_7 & b_8\\
b_2 & b_1 & -b_4 & b_3 & b_6 & -b_5 & -b_8 & b_7\\
-b_3 & b_4 & b_1 & b_2 & -b_7 & b_8 & -b_5 & b_6\\
-b_4 & -b_3 & -b_2 & b_1 & b_8 & b_7 & b_6 & b_5\\
b_5 & -b_6 & b_7 & -b_8 & b_1 & b_2 & -b_3 & b_4\\
b_6 & b_5 & -b_8 & -b_7 & -b_2 & b_1 & b_4 & b_3\\
-b_7 & b_8 & b_5 & -b_6 & b_3 & -b_4 & b_1 & b_2\\
-b_8 & -b_7 & -b_6 & -b_5 & -b_4 & -b_3 & -b_2 & b_1
\end{pmatrix}.$$

We can see that $H(\phi\alpha)$ has the structure of a Hurwitz matrix (see, for example [8]).

In the case of Kustaanheimo–Stiefel transformation $(3 \to 4)$ the inverse form is called Cayley–Klein [9]. For the case of Hurwitz transformation $(5 \to 8)$ the analogical form was first given in [12]. According to our knowledge the transformation (6) for Hurwitz transformation $(9 \to 16)$ is established in the present paper for the first time. This explicit inverse form is very convenient to use for concrete calculations as shown in the following sections.

3. Connection between Schrödinger equations

Let us consider the Schrödinger equation of a 16-dimensional isotropic harmonic oscillator

$$\left\{-\frac{1}{8}\left(\frac{\partial^2}{\partial u_s \partial u_s} + \frac{\partial^2}{\partial v_s \partial v_s}\right) - \frac{1}{2}\omega^2(u_s u_s + v_s v_s)\right\}\Psi(u, v) = Z\,\Psi(u, v). \tag{7}$$

Here ω, Z have positive real values and are correspondingly the frequency and energy of the harmonic oscillator. After directly applying the transformation (6) to equation (7), using (3) we have

$$\left\{-\frac{1}{2}\frac{\partial^2}{\partial x_\lambda \partial x_\lambda} - \frac{1}{2}(\Gamma_j)_{st} v_t \frac{\partial \phi_k}{\partial u_s}\frac{\partial^2}{\partial x_j \partial \phi_k} - \frac{1}{8}\frac{\partial \phi_k}{\partial u_s}\frac{\partial \phi_j}{\partial u_s}\frac{\partial^2}{\partial \phi_k \partial \phi_j} - \frac{1}{8}\frac{\partial^2 \phi_k}{\partial u_s \partial u_s}\frac{\partial}{\partial \phi_k} - \frac{Z}{r}\right\}\Psi(\mathbf{r}, \phi\alpha) = E\,\Psi(\mathbf{r}, \phi\alpha), \tag{8}$$

where $E = -\omega^2/2$ is a negative number that denotes the energy of bound states; Z becomes a parameter defining the 'charge' value in the Coulomb potential; for simplicity in notation we use (only here) ϕ_k $(k = 1, 2, \ldots, 7)$ to indicate the angles $\phi_1, \phi_2, \phi_3, \phi_4, \varphi_1, \varphi_2, \varphi_3$.

For the wavefunctions independent of the angles $\phi\varphi$, say $\Psi(\mathbf{r})$, equation (8) becomes the Schrödinger equation for a hydrogen-like atom in nine-dimensional real space. It means that among the wavefunctions of a 16-dimensional harmonic oscillator we can extract the wavefunctions that satisfy the following seven conditions:

$$\frac{\partial \Psi}{\partial \varphi_1} = 0, \quad \frac{\partial \Psi}{\partial \varphi_2} = 0, \quad \frac{\partial \Psi}{\partial \varphi_3} = 0,$$
$$\frac{\partial \Psi}{\partial \phi_1} = 0, \quad \frac{\partial \Psi}{\partial \phi_2} = 0, \quad \frac{\partial \Psi}{\partial \phi_3} = 0, \quad \frac{\partial \Psi}{\partial \phi_4} = 0, \tag{9}$$

and they are the wavefunctions of a hydrogen-like atom in nine-dimensional space.

An interesting question appears regarding what other physical meaning is contained in the wavefunctions of a 16-dimensional harmonic oscillator that do not satisfy conditions (9). The answer will be provided in the following section.

4. Hidden non-Abelian monopole

By substituting the explicit expression (6) of the extra variables into equation (8) we can rewrite it in the new form as follows:

$$\left\{\frac{1}{2}\left(-\mathrm{i}\frac{\partial}{\partial x_\lambda} + A_{\lambda k}(\mathbf{r})\hat{I}_{k\bar{\lambda}}(\phi\varphi)\right)\left(-\mathrm{i}\frac{\partial}{\partial x_\lambda} + A_{\lambda k}(\mathbf{r})\hat{I}_{k\bar{\lambda}}(\phi\varphi)\right) + \frac{1}{2r}\hat{I}^2(\phi\varphi) - \frac{Z}{r}\right\}\Psi(\mathbf{r}, \phi\alpha) = E\,\Psi(\mathbf{r}, \phi\alpha) \tag{10}$$

with the septet of potential vectors $A_{\lambda k}(\mathbf{r})(\lambda = 1, 2, \ldots, 9; k = 1, 2, \ldots, 7)$

$$\begin{aligned}
A_{\lambda 1} &= \frac{\mathrm{i}}{2r(r+x_9)}(-x_2, x_1, x_4, -x_3, x_6, -x_5, x_8, -x_7, 0)\\
A_{\lambda 2} &= \frac{\mathrm{i}}{2r(r+x_9)}(x_3, x_4, -x_1, -x_2, x_7, -x_8, -x_5, x_6, 0)\\
A_{\lambda 3} &= \frac{\mathrm{i}}{2r(r+x_9)}(x_4, -x_3, x_2, -x_1, x_8, x_7, -x_6, -x_5, 0)\\
A_{\lambda 4} &= \frac{\mathrm{i}}{2r(r+x_9)}(-x_5, -x_6, -x_7, x_8, x_1, x_2, x_3, -x_4, 0)\\
A_{\lambda 5} &= \frac{\mathrm{i}}{2r(r+x_9)}(x_6, -x_5, x_8, x_7, x_2, -x_1, -x_4, -x_3, 0)\\
A_{\lambda 6} &= \frac{\mathrm{i}}{2r(r+x_9)}(x_7, -x_8, -x_5, -x_6, x_3, x_4, -x_1, x_2, 0)\\
A_{\lambda 7} &= \frac{\mathrm{i}}{2r(r+x_9)}(-x_8, -x_7, x_6, x_5, -x_4, -x_3, x_2, x_1, 0).
\end{aligned} \tag{11}$$

Table 1. Expression of operators $\hat{I}_{k\lambda}(\phi\varphi)$.

	1	2	3	4	5	6	7
1	$J_1^0 + J_2^0$	$J_1^+ + J_2^+$	$J_1^- - J_2^-$	$-Q_1^+ - Q_2^+$	$Q_1^- - Q_2^-$	$K_1^+ + K_2^+$	$K_1^- - K_2^-$
2	$J_1^0 + J_2^0$	$J_1^+ - J_2^+$	$J_1^- + J_2^-$	$-Q_1^+ + Q_2^+$	$Q_1^- + Q_2^-$	$K_1^+ - K_2^+$	$K_1^- + K_2^-$
3	$J_1^0 - J_2^0$	$J_1^+ + J_2^+$	$J_1^- + J_2^-$	$Q_3^+ + Q_4^+$	$Q_3^- + Q_4^-$	$K_3^+ - K_4^+$	$K_3^- - K_4^-$
4	$J_1^0 - J_2^0$	$J_1^+ - J_2^+$	$J_1^- - J_2^-$	$Q_3^+ - Q_4^+$	$Q_3^- - Q_4^-$	$K_3^+ + K_4^+$	$K_3^- + K_4^-$
5	$J_3^0 + J_4^0$	$J_3^+ + J_4^+$	$J_3^- - J_4^-$	$-Q_1^+ - Q_2^+$	$Q_1^- + Q_2^-$	$K_3^+ - K_4^+$	$K_3^- + K_4^-$
6	$J_3^0 + J_4^0$	$J_3^+ - J_4^+$	$J_3^- + J_4^-$	$-Q_1^+ + Q_2^+$	$Q_1^- - Q_2^-$	$K_3^+ + K_4^+$	$K_3^- - K_4^-$
7	$J_3^0 - J_4^0$	$J_3^+ + J_4^+$	$J_3^- + J_4^-$	$Q_3^+ + Q_4^+$	$Q_3^- - Q_4^-$	$K_1^+ + K_2^+$	$K_1^- + K_2^-$
8	$J_3^0 - J_4^0$	$J_3^+ - J_4^+$	$J_3^- - J_4^-$	$Q_3^+ - Q_4^+$	$Q_3^- + Q_4^-$	$K_1^+ - K_2^+$	$K_1^- - K_2^-$

In equation (10) we have 56 operators $\hat{I}_{k\tilde{\lambda}}(\phi\varphi)(\lambda = 1, 2, \ldots, 8; k = 1, 2, \ldots, 7)$ but only 28 of them are different. Here and further, the symbol $\tilde{\lambda}$ with the $\sim$ means the repeating index that does not indicate the summation over them. For convenience we will define 28 other operators $\hat{J}_s^0$, $\hat{J}_s^{\pm}$, $\hat{Q}_s^{\pm}$,$\hat{K}_s^{\pm}$ $(s = 1, 2, 3, 4)$ and express $\hat{I}_{k\tilde{\lambda}}(\phi\varphi)$ via them as shown in table 1.

The 28 operators $\hat{J}_s^0$, $\hat{J}_s^{\pm}$, $\hat{Q}_s^{\pm}$,$\hat{K}_s^{\pm}$ set up a closed algebra as will be shown in the following section, where the explicit form of these operators and their commutation relations will be given.

It is easy to verify that the septet of potential vectors (11) holds the following properties:

$$x_\lambda A_{\lambda k} = 0, \qquad A_{\lambda j} A_{\lambda k} = \delta_{jk} \frac{r - x_9}{4r^2(r + x_9)} \tag{12}$$

for every value $k = 1, 2, \ldots, 7$ and has a singularity line along the negative part of axis ox_9. We can say that each term of (11) concerns the nine-dimensional vector potential of the monopole. Indeed, remember the case of the Dirac monopole [10] where the potential vector has the form

$$A_\lambda = \frac{\mathrm{i}}{2r(r + x_3)}(-x_2, x_1, 0)$$

and satisfies a condition like (12):

$$x_\lambda A_\lambda = 0, \qquad A_\lambda A_\lambda = \frac{r - x_3}{4r^2(r + x_3)}.$$

Analogically, in the case of a non-Abelian SU(2) model [13], a monopole field is described by a triplet of potential vectors

$$A_{\lambda 1} = \frac{1}{r(r - x_5)}(-x_2, x_1, x_4, -x_3, 0)$$
$$A_{\lambda 2} = \frac{1}{r(r - x_5)}(x_3, x_4, -x_1, -x_2, 0)$$
$$A_{\lambda 3} = \frac{1}{r(r - x_5)}(x_4, -x_3, x_2, -x_1, 0)$$

that satisfy the condition similar to (12) and associate with three operators of SU (2) algebra. Thus, in our case, we deal with a monopole described by a septet of potential vectors (11) associating with 28 operators of a closed algebra. By dimension, the SO$(2n)$ algebra has $(2n - 1) \times n$ elements equal to 28 in the case of $n = 4$ that suggests to us to consider the system described by (11) with 28 operators $\hat{J}_s^0$, $\hat{J}_s^{\pm}$, $\hat{Q}_s^{\pm}$,$\hat{K}_s^{\pm}$ as a non-Abelian SO (8) model of the monopole. However it needs additional investigation, which we will do in our next works.

5. The algebra

In this section we will give the explicit form of 28 operators $\hat{J}_s^0$, $\hat{J}_s^{\pm}$, $\hat{Q}_s^{\pm}$,$\hat{K}_s^{\pm}$ and their commutation relations.

First of all, we define three operators

$$\begin{aligned}
\hat{J}_1^0 &= \frac{\partial}{\partial\alpha_1} + \frac{\partial}{\partial\alpha_2}\\
\hat{J}_1^+ &= \sin(\alpha_1+\alpha_2)\left(\tan(\phi_2/2)\frac{\partial}{\partial\alpha_1} - \cot(\phi_2/2)\frac{\partial}{\partial\alpha_2}\right) + 2\cos(\alpha_1+\alpha_2)\frac{\partial}{\partial\phi_2}\\
\hat{J}_1^- &= \cos(\alpha_1+\alpha_2)\left(\tan(\phi_2/2)\frac{\partial}{\partial\alpha_1} - \cot(\phi_2/2)\frac{\partial}{\partial\alpha_2}\right) - 2\sin(\alpha_1+\alpha_2)\frac{\partial}{\partial\phi_2}
\end{aligned}\tag{13}$$

which have such a form that $\alpha_1, \alpha_2, \phi_2$ can be considered as three Euler angles. The other operators $\hat{J}$ can be obtained from operators (13) too. Particularly, operators $\hat{J}_2^0$, $\hat{J}_2^+$, $\hat{J}_2^-$ are defined from (13) by changing $\alpha_1 \to \alpha_3$, $\alpha_2 \to \alpha_4$, $\phi_2 \to \phi_3$; J_3^0, J_3^+, J_3^- and J_4^0, J_4^+, J_4^- are from J_1^0, J_1^+, J_1^- and J_2^0, J_2^+, J_2^- just by changing $\alpha_2 \to -\alpha_2$ and $\alpha_4 \to -\alpha_4$ correspondingly. It is easy to verify that 12 operators $\hat{J}_s^0$, $\hat{J}_s^+$, $\hat{J}_s^-(s = 1, 2, 3, 4)$ satisfy the following commutation relations:

$$\begin{aligned}
&[J_s^-, J_t^-] = 0, \qquad &&[J_s^+, J_t^+] = 0\\
&[J_s^+, J_t^-] = 2\delta_{s\tilde{t}}J_t^0, \qquad &&[J_s^0, J_t^{\pm}] = \pm 2\delta_{s\tilde{t}}J_t^{\mp}.
\end{aligned}\tag{14}$$

Now let us define the operators $\hat{Q}$, the first of which have the explicit form:

$$\begin{aligned}
\hat{Q}_1^+ &= \sin(\alpha_1+\alpha_3)\left(-\frac{\tan(\phi_1/2)\cos(\phi_3/2)}{\cos(\phi_2/2)}\frac{\partial}{\partial\alpha_1} + \frac{\cot(\phi_1/2)\cos(\phi_2/2)}{\cos(\phi_3/2)}\frac{\partial}{\partial\alpha_3}\right)\\
&\quad - 2\cos(\alpha_1+\alpha_3)\left(\cos(\phi_2/2)\cos(\phi_3/2)\frac{\partial}{\partial\phi_1} + \sin(\phi_2/2)\cos(\phi_3/2)\tan(\phi_1/2)\frac{\partial}{\partial\phi_2}\right.\\
&\quad \left. - \cos(\phi_2/2)\sin(\phi_3/2)\cot(\phi_1/2)\frac{\partial}{\partial\phi_3}\right)\\
\hat{Q}_1^- &= \cos(\alpha_1+\alpha_3)\left(-\frac{\tan(\phi_1/2)\cos(\phi_3/2)}{\cos(\phi_2/2)}\frac{\partial}{\partial\alpha_1} + \frac{\cot(\phi_1/2)\cos(\phi_2/2)}{\cos(\phi_3/2)}\frac{\partial}{\partial\alpha_3}\right)\\
&\quad + 2\sin(\alpha_1+\alpha_3)\left(\cos(\phi_2/2)\cos(\phi_3/2)\frac{\partial}{\partial\phi_1} + \tan(\phi_1/2)\sin(\phi_2/2)\cos(\phi_3/2)\frac{\partial}{\partial\phi_2}\right.\\
&\quad \left. - \cot(\phi_1/2)\cos(\phi_2/2)\sin(\phi_3/2)\frac{\partial}{\partial\phi_3}\right).
\end{aligned}\tag{15}$$

The other operators $\hat{Q}$ can be obtained from (15) too. Indeed, $\hat{Q}_2^+$, $\hat{Q}_2^-$ are from $-\hat{Q}_1^+$, $-\hat{Q}_1^-$ by changing $\alpha_1 \to \alpha_2, \alpha_3 \to \alpha_4, \phi_2 \to \pi - \phi_2, \phi_3 \to \pi - \phi_3$ whereas $\hat{Q}_3^+$, $\hat{Q}_3^-$ and $\hat{Q}_4^+$, $\hat{Q}_4^-$ are from $\hat{Q}_1^+$, $-\hat{Q}_1^-$ and $\hat{Q}_2^+$, $\hat{Q}_2^-$ by changing $\alpha_3 \to -\alpha_3$ and $\alpha_4 \to -\alpha_4$, respectively. Eight operators $\hat{Q}_s^+$, $\hat{Q}_s^-(s = 1, 2, 3, 4)$ together with four operators $\hat{J}_s^0(s = 1, 2, 3, 4)$ construct a close algebra because they satisfy the following commutation relations:

$$\begin{aligned}
&[\hat{Q}_s^+, \hat{Q}_t^+] = 0, \qquad &&[\hat{Q}_s^-, \hat{Q}_t^-] = 0\\
&[\hat{Q}_s^+, \hat{Q}_t^-] = \delta_{st}\alpha_{\tilde{s}q}\hat{J}_q^0, \qquad &&[\hat{J}_s^0, \hat{Q}_t^{\pm}] = \pm\alpha_{s\tilde{t}}\hat{Q}_t^{\mp},
\end{aligned}\tag{16}$$

where the structure constants α_{st} are elements of the matrix

$$\alpha = \begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

We will define also eight operators $K_s^+, K_s^-(s = 1, 2, 3, 4)$ as follows: K_1^+, K_1^- from Q_1^+, Q_1^- by changing $\alpha_2 \to \alpha_1, \phi_2 \to \pi - \phi_2$; K_2^+, K_2^- from K_1^+, K_1^- by changing $\alpha_2 \to \alpha_1$, $\alpha_3 \to \alpha_4, \phi_2 \to \pi - \phi_2, \phi_3 \to \pi - \phi_3$; K_3^+, K_3^- from K_1^+, K_1^- by changing $\alpha_3 \to -\alpha_3$; K_4^+, K_4^- from K_2^+, K_2^- by changing $\alpha_4 \to -\alpha_4$. These operators satisfy

$$\begin{aligned} &\left[K_s^+, K_t^+\right] = 0, && \left[K_s^-, K_t^-\right] = 0 \\ &\left[K_s^+, K_t^-\right] = (-1)^{\delta_{3t}} \delta_{st} \alpha_{\tilde{s}q} J_q^0, && \left[J_s^0, K_t^\pm\right] = \pm(-1)^{\delta_{3t}} \alpha_{s\tilde{t}} K_t^\mp. \end{aligned} \tag{17}$$

Consequently, we have 28 operators including $J_s^0, J_s^\pm, Q_s^\pm, K_s^\pm$ $(s = 1, 2, 3, 4)$ that construct a closed algebra. Indeed, besides (14), (16), (17) additionally these operators satisfy the following commutation relations:

$$\begin{aligned} &\left[J_s^+, Q_t^+\right] = K^+_{\{s+(-1)^{s+1}(t+1)\}}, && \left[J_s^-, Q_t^-\right] = (-1)^{\delta_{3t}} \alpha_{\tilde{s}\tilde{t}} K^+_{\{s+(-1)^{s+1}(t+1)\}}, \\ &\left[Q_s^+, K_t^+\right] = J^+_{\{s+(-1)^{s+1}(t+1)\}}, && \left[Q_s^-, K_t^-\right] = (-1)^{\delta_{3t}} \alpha_{\tilde{s}\tilde{t}} J^+_{\{s+(-1)^{s+1}(t+1)\}}, \\ &\left[K_s^+, J_t^+\right] = Q^+_{\{s+(-1)^{s+1}(t+1)\}}, && \left[K_s^-, J_t^-\right] = (-1)^{\delta_{3t}} \alpha_{\tilde{s}\tilde{t}} Q^+_{\{s+(-1)^{s+1}(t+1)\}}, \\ &\left[J_s^+, Q_t^-\right] = (-1)^{\delta_{1s}+\delta_{3t}} K^-_{\{s+(-1)^{s+1}(t+1)\}}, && \left[Q_s^-, K_t^+\right] = -(-1)^{\delta_{1s}} \alpha_{\tilde{s}\tilde{t}} J^-_{\{s+(-1)^{s+1}(t+1)\}}, \\ &\left[Q_s^+, K_t^-\right] = (-1)^{\delta_{3t}+\delta_{1s}} J^-_{\{s+(-1)^{s+1}(t+1)\}}, && \left[K_s^-, J_t^+\right] = -(-1)^{\delta_{1s}} \alpha_{\tilde{s}\tilde{t}} Q^-_{\{s+(-1)^{s+1}(t+1)\}}, \\ &\left[K_s^+, J_t^-\right] = (-1)^{\delta_{3t}+\delta_{1s}} Q^-_{\{s+(-1)^{s+1}(t+1)\}}, && \left[J_s^-, Q_t^+\right] = -(-1)^{\delta_{1s}} \alpha_{\tilde{s}\tilde{t}} K^-_{\{s+(-1)^{s+1}(t+1)\}}. \end{aligned} \tag{18}$$

Here δ_{st} is Kronecker delta and the index $n = \{s + (-1)^{s+1}(t + 1)\}$ should take an integer value of 1, 2, 3, 4 only. With the indices s, t having the value of 1, 2, 3, 4 the formula $s + (-1)^{s+1}(t + 1)$ gives an integer number from −3 to +8 out of the index range. Therefore, we use the circular rule denoted by the bracket { } to relate them with the index value 1, 2, 3, 4 as follows:

$$\begin{aligned} &\{-3\} = 1, && \{-2\} = 2, && \{-1\} = 3, && \{0\} = 4, \\ &\{1\} = 1, && \{2\} = 2, && \{3\} = 3, && \{4\} = 4, \\ &\{5\} = 1, && \{6\} = 2, && \{7\} = 3, && \{8\} = 4. \end{aligned}$$

6. Summary and outlook

We have successfully built the generalized Hurwitz transformation by adding seven extra new variables. The inverse transformation thus has been established. An interesting result has been achieved in the wavefunctions of a 16-dimensional harmonic oscillator hidden in the solutions of a nine-dimensional ‘charged-dyon’ system of a non-Abelian model. Although the explicit algebra associated with the model has been given, detailed investigation needs to be performed in the near future. The connection established between two fundamental problems such as a 16-dimensional harmonic oscillator and the nine-dimensional ‘charge-dyon’ system gives us the simple way to investigate the dynamical symmetry of the latter. That is analogical to what we have successfully done for the case of lower dimensions [12]. We will return to this problem in our next work.

References

[1] Davtyan L S, Mardoyan L G, Pogosyan G S, Sissakian A N and Ter-Antonyan V M 1987 *J. Phys. A: Math. Gen.* **20** 6121

[2] Lambert D and Kibler M 1988 *J. Phys. A: Math. Gen.* **21** 307

[3] Kleinert H 1983 *Found. Phys.* **23** 5
Kleinert H 1968 *Lectures in Theoretical Physics* XB ed W E Brezin and A O Barut (New York: Gordon and Brack) 427

[4] Le V H and Komarov L I 1993 *Phys. Lett.* A **177** 121
[5] Hurwitz A 1896 *Nachr. Ges. Wiss. Gottingen* 309
[6] Levi-Civita T 1956 *Opere Mat.* **2** 1901
[7] Kustaanheimo P and Stiefel E 1965 *J. Reine Angrew. Math.* **218** 204
[8] Davtyan L S, Sissakian A N and Ter-Antonyan V M 1995 *J. Math. Phys.* **36** 664
[9] Kibler M and Negadi T 1983 *Croatica Chem. Acta* **57** 1509
[10] Kleinert H 1986 *Phys. Lett.* A **116** 201
[11] Le V H and Viloria T J 1992 *Phys. Lett.* A **171** 23
[12] Le V H, Viloria T J and Le A T 1991 *J. Phys. A: Math. Gen.* **24** 3021
[13] Mardoyan L G, Sissakian A N and Ter-Antonyan V M 1996 arXiv: hep-th/9601093
Mardoyan L G, Sissakian A N and Ter-Antonyan V M 1999 *Mod. Phys. Lett.* A **14** 1303
Nersessian A and Pogosyan G 2001 *Phys. Rev.* A **63** 020103
[14] Pletyukhov T V and Tolkachev E A 1999 *J. Phys. A: Math. Gen.* **32** L249
[15] Karayn Kh H, Mardoyan L G and Ter-Antonyan V M 2003 *J. Cont. Phys.* **38** 78
Mardoyan L G, Pogosyan G S and Sissakian A N 2004 *Superintegrability in Classical and Quantum Systems* ed P Tempesta (Providence, RI: AMS Bookstore) p 155